**BREAKING DOWN THE BARRIERS**

Though similarly path-dependent, barrier options are easier to value than standard American options, as Mark Rubinstein and Eric Reiner explain.

The pay-off of a standard European option depends only on the price of the underlying asset on the expiration date. In particular, given the final price of the underlying asset, the pay-off will be the same regardless of the path taken by the underlying asset during the life of the option to reach that final price. Whether the underlying asset price reaches a given price by moving down and then up, or up and then down, matters not to the buyer or seller of the option. It is as if it did not matter whether you travelled from Paris to London by air or by the Channel Tunnel, as long as you arrived on time.

The terminology used to describe this feature is path independence. More generally, the pay-off from an option may depend on some aspect of the price path. For example, the pay-off of a lookback option depends on the minimum or maximum price of the underlying asset attained during the life of the option, and the pay-off of an Asian option depends on the average price. In this article, we will examine a simpler type of path-dependent option in which the pay-off depends not only on the final price of the underlying asset but also on whether or not the underlying asset has reached some other barrier price during the life of the option.

Our objective is to value a variety of these options in a Black-Scholes environment, that is, in one in which the underlying asset can be assumed to follow a log-normal random walk, and in which arbitrage arguments allow the use of a risk-neutral valuation approach—discount the expected pay-off of the option at expiration by the riskless rate, where the underlying asset price is expected to appreciate at the same riskless rate less payouts.

These options are in a sense intermediate between standard European and American options. They resemble American options since their value depends on how the underlying asset price behaves through time. But they are simpler to value than American options since the critical boundary of the underlying asset price is determined in advance and specified in the contract. As a result, unlike American options, it is possible to state "closed form" valuation solutions.

To do this we will need the density of the natural logarithm of the risk-neutral underlying asset return, \( u \):

\[
\frac{1}{\pi} \sqrt{e^{-2\sigma^2}} \exp(-uv)
\]

with \( u = (\mu - r)/\sigma \sqrt{t}, \mu = \log(S_0) - 1/2 \sigma^2 \)

This is just a normal density function in which \( r \) is one plus the rate of interest, \( \text{d} \) is one plus the pay-out rate of the underlying asset, \( \sigma \) is the volatility of the underlying asset, and \( t \) is the time-to-expiration of the option.

We also need another density. Given that the underlying asset price first starts at \( S_0 \) above the barrier \( H \), the density of the natural logarithm of the underlying asset return when the underlying asset price breaches the barrier but ends up below the barrier at expiration is:

\[
\frac{1}{\pi} \sqrt{e^{-2\sigma^2}} \exp(-uv)
\]

with \( u = (\mu - r - \nu)/\sigma \sqrt{t}, \nu = \log(H/S) \)

This is a normal density premultiplied by \( e^{\nu u} \). Here \( \eta = 1 \). Alternatively, given that the underlying asset price first starts below the barrier, the density of the natural logarithm of the underlying asset return when the underlying asset price breaches the barrier but ends up at expiration below the barrier is the same expression but in which \( \eta = -1 \).

"In" barrier options

Our first example is a down-and-in call. Although you pay for this option up front, you do not receive the call until the underlying asset price reaches a prespecified level termed the barrier or knock-in boundary. \( H \). If, after elapsed time \( t \leq \tau \), the underlying asset price hits the barrier, you receive a standard European call with strike price \( K \) and time-to-expiration \( \tau - t \). On the other hand, if through

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1. To our knowledge, the only published solution for the options covered in this article has been for the down-and-out option (without a guaranteed for payout). See John Cox and Mark Rubinstein, Options Markets, page 410, Prentice-Hall, 1985.
2. Since Black and Scholes are only interested in the price of the underlying asset at expiration, they can have \( \nu, \sigma \), and \( H \) to be known numbers at time zero. However, since the options depend on complex ways on the time paths of these variables, to keep matters simple we assume here that these variables are constant through time.
3. In this case, the standard call is received conditional on the behaviour of a random variable (underlying asset price). See, Mark Rubinstein, Pay-non, exercise later, Risk, February 1991, for an analysis of a standard call that is received unconditionally but at some prespecified future date.
Barrier options are simpler to value than American options since the critical boundary of the underlying asset price is determined in advance and specified in the contract. It is therefore possible to state "closed form" valuation solutions.
For the $K>H$ case, the new quantities are $\text{prob}(S^*<H)$ and $\text{prob}(S^*>H; S(t)=H)$. The density corresponding to the former is of course ful and the density corresponding to the latter is identical to $q(u)$, but with $\eta = -1$. Therefore,

$$C_{K>H} = [1] + [5] (\eta = -1, \phi = 1)$$

For the $K<H$ case, we first restate

$$\text{prob}(H>S^*>K; S(t)\geq H) = \text{prob}(S^*<H; S(t)\geq H)$$

Then, we can write immediately that


For our next options, down-and-in puts and up-and-in puts, we simply provide graphs (Figures 3 and 4) from which the stated results can easily be inferred.

"Out" barrier options

Corresponding to each of these four "in" barrier options are four "out" options. For example, in a down-and-out call, a standard call comes into existence when the down-and-out is issued, but the standard call is extinguished prior to expiration if the underlying asset price ever drops below the knock-out boundary, $H$. In that case the buyer of the option may be paid a fixed rebate, $R$. Otherwise, if the underlying asset price never drops below $H$, the down-and-out call will have the same pay-off as a standard call. Expressed concisely, the pay-off from this option is:

$$\text{down-and-out call max}(0, S^*-K) \text{ if for all } t, S(t) > H$$

$$S(t) \geq H \text{ at hit} \text{ if for some } t \leq T, S(t) \leq H$$

Here is one possible use of a down-and-out call. Suppose you are holding a covered call but will be forced to liquidate the underlying asset if its price falls sharply. If you sold a down-and-out call in place of a standard call, you could arrange to have the call liquidated automatically at the same time.

If the rebate $R=0$, the following parity relationship makes it easy to write down the values of down-and-out calls:

$$\text{pay-off from standard option} = \text{pay-off from down-and-out option} + \text{pay-off from down-and-in option}$$

To see this, suppose you own otherwise identical down-and-out and down-and-in options with no rebates. If the common barrier is never hit, then you receive the


$$P_{u(K,H)} = [1] + [5] (\eta = 1, \phi = -1)$$
γ-off from a standard option; if the
munition barrier is hit, as the down-and-
top option is extinguished, the down-and-
option delivers you a standard option
initial to the one you lost when the
up-and-down option was cancelled.
us, even in this case, you end up receiv-
ing the pay-off from a standard option.
The only difficulty comes from the re-
view. For "in" options, it is not possible
receive the rebate prior to expiration,
nor one continues to remain in doubt
whether or not the barrier will never
hit. However, to an "out" option, it
possible as well as customary for the
rate to be paid the moment the barrier
hit. This complicates the risk-neutrali-
zation problem since the rebate may
be received at a random rather than
specified time. Thus, we need an addi-
tional density of the first passage time (t)
the underlying asset price to hit the
barrier:
$$b(t) = (-\eta a / a^2 + \eta b) e^{-t}$$
with $\eta = -\eta_0 + \eta_0 t / \sqrt{\tau}$

Here, $\eta = 1$ if the barrier is being
approached from above and $\eta = -1$ if
the barrier is being approached from be-
low. The present value of the rebate is
then the expected rebate discounted by
the interest rate raised to the power of
the first passage time:
$$[6] = R \int_{t=0}^{\infty} \left[ e^{-rt} \right] dt = R \left[ (H/S)^{\eta(t)} - \eta_0 t \right]$$
where the region of integration is from $0$
to $t$, and
$$z = \log(H/S) + \alpha \sqrt{t} + a \sqrt{t}$$
$$a = \mu / \sigma, b = [\sqrt{\mu^2 + 2 \log(\sigma^2)} / \sigma^2]$$

Using these relationships, we can now
rewrite the valuation solutions for the
down-and-out call and the three remain-
ing "out" options:
$$C_{s_h < H} = [1 - \eta] + [5] (\eta = -1, \phi = 1)$$
$$C_{s_h < H} = [2 - \eta] + [5] (\eta = 1, \phi = 1)$$

At first it may be surprising that the rebate
provides the only contribution to the value
of an up-and-out call when the strike price
is greater than the barrier. But it is easy
to see why. Since $S < H < K$, in order for
the underlying asset price to end up above
the strike price it must first breach the
barrier, but in this event, the call is extin-
guished. Similarly, a down-and-out put
will also only be valued for the rebate
when the strike price is less than the
barrier.

**Figure 4**

**up-and-in put:**
max(0, K - S) if for some $t < T$, $S(t) \geq H$
R (at expiry) if for all $t < T$, $S(t) < H$

**Put pay-off (K<H):**
prob([H≤S<K) + prob([S<CH: S(t)≥H]

**Put pay-off (K>h):**
prob(S<rH: S(t)≥H) - prob(S<rH: S(t)<H]

\[
P_{\text{up}(H)} = [1 - \eta] + [4] + [5] (\eta = -1, \phi = -1)
\]
\[
P_{\text{up}(K)} = [3] + [5] (\eta = -1, \phi = -1)
\]

Mark Rubinstein is a professor of finance at the University of California, Berkeley, and he and Eric Ronen are,
respectively, a principal and vice-president of Leland
O'Brien Rubinstein Associates.