DRAWDOWN-BASED STOP-OUTS AND THE “TRIPLE PENANCE” RULE

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ABSTRACT

We develop a framework for informing the decision of stopping a portfolio manager or investment strategy once it has reached the drawdown or time under water limit associated with a certain confidence level. Under standard portfolio theory assumptions, we show that it takes three times longer to recover from the maximum drawdown than the time it took to produce it, with the same confidence level (“triple penance rule”).

We provide a theoretical justification to why investment firms typically set less strict stop-out rules to portfolio managers with higher Sharpe ratios, despite the fact that they should be expected to deliver superior performance. We generalize this framework to the case of first-order auto-correlated investment outcomes, and conclude that ignoring the effect of serial correlation leads to a gross underestimation of the downside potential of hedge fund strategies, by as much as 70%. We also estimate that some hedge funds may be firing more than three times the number of skillful portfolio managers, compared to the number that they were willing to accept, as a result of evaluating their performance through traditional metrics, such as the Sharpe ratio.

We believe that our closed-formula compact expression for the estimation of drawdown potential, without having to assume IID cashflows, will open new practical applications in risk management, portfolio optimization and capital allocation. The Python code included confirms the accuracy of our solution.

Keywords: Drawdown, time under water, stop-out, triple penance, serial correlation, Sharpe ratio.

JEL Classification: G0, G1, G2, G15, G24, E44.
1. INTRODUCTION
Multi-manager investment firms are routinely faced with the decision to stop-out a portfolio manager (PM). This is a decision of the utmost importance, intended to protect the well-being of the overall funds. It typically has dramatic consequences, including the removal of the PM involved. Despite of the relevance and recurrence of stop-outs, we are not aware of the existence of a theoretical framework addressing this particular question. Such framework would be very useful in practice, as it would allow the firm to approach the stop-out problem in an objective and transparent manner, thus avoiding the personal conflict that is so often associated with employee dismissals.

The decision to discontinue (or stop-out) an investment strategy can be approached as a decision problem under uncertainty. This is because we cannot be certain whether negative outcomes, accumulated over time, are the result of bad performance, or mere bad luck. In this paper we present a framework that answers the question of stopping out an investment in two complementary formulations: Cumulative loss from the previous high-watermark (drawdown, or DD) and time elapsed since the previous high-watermark (time under water, or TuW).

There are three prominent definitions of drawdown in the academic literature: (i) Drawdown as a quantile (an analogue to VaR), as in López de Prado and Peijan [2004], Mendes and Leal [2005] or Hayes [2006]; (ii) Drawdown as an extreme value, as in Magdon-Ismail and Atiya [2004]; and (iii) Drawdown as the average of a specified percentage of the largest drawdowns over an investment horizon (as an analogue to CVaR), as in Chekhlov, Uryasev and M. Zabarankin [2003, 2005]. The three definitions are useful, and our choice for a quantile-based drawdown is inspired by the greater acceptance that VaR has among practitioners.

The literature has discussed drawdown measures since at least the groundbreaking work of Grossman and Zhou [1993], who analyzed the optimal risky investment policy in continuous-time for an investor who, at each point in time, is willing to lose no more than a fixed percentage of the maximum value his wealth has achieved up to that time. Their work shared some resemblance with the “Constant Proportion Portfolio Insurance” (CPPI) policies which were popular at that time (Meucci [2010]). Their approach considered the continuous-rebalancing of the investment on a single risky asset, and Cvitanic and Karatzas [1995] discussed a framework with multiple assets in continuous-time. More recently, Yang and Zhong [2012] extended this result to a discrete-time framework, which they call “Rolling Economic Drawdown.” Alexander and Baptista [2006] provide a characterization of mean-variance optimal portfolios subject to a drawdown constraint, and find that this constraint affects negatively a PM’s ability to track a benchmark. Chekhlov, Uryasev and M. Zabarankin [2003, 2005] introduce a one-parameter family of risk measures called Conditional Drawdown-at-Risk (CDaR), and solve a mean return-CDaR optimization problem. Cherny and Obloj [2011] solve a portfolio optimization problem subject to non-linear drawdown constraints in a semimartingale financial model. All of these works assumed random walk return dynamics, and attempted to compute the exposures that maximize a utility function (a portfolio choice problem).
On the risk side, there have also been notable contributions. Magdon-Ismail and Atiya [2004] and Magdon-Ismail, Atiya et al. [2004] determine the expected maximum drawdown for a Brownian motion with drift. This differs from previous studies in that drawdown is defined as a supremum, rather than a quantile or a conditional expectation. Mendes and Leal [2005] proposed an extreme value distribution for modeling the maximum drawdown, and studied the empirical relation between drawdowns and fractionally integrated GARCH volatility on Nasdaq and FTSE series. Pavlikov, Uryasev and Zabarankin [2012] adapt the CAPM framework to incorporate a drawdown alpha and a drawdown beta, which allows them to rank hedge fund strategies and identify instruments for hedging against market drawdowns. Carr, Zhang and Hadjiliadis [2011] introduce an insurance against a large realization of maximum drawdown and a novel way to hedge the liability incurred in underwriting this insurance.

In this paper we will study the question of downside potential from an investment management perspective. Consider a multi-manager firm, whose executives must decide where (DD) and when (TuW) a particular PM must be stopped-out.¹ Multi-manager firms allocate capital to a PM based on the statistics that characterize his track record. These statistics are then used to generate DD and TuW limits, which reflect the firm’s appetite for false positives. Very tight DD and TuW limits are adopted by firms who are willing to fire a truly skillful PM, however unlucky he may have been. In other words, realized DD and TuW exceeding certain limits are taken as sufficient evidence (for a certain confidence level) that the PM is not living up to the expectations which granted him a capital allocation.

A PM’s compensation is typically a function of performance over a previous high-watermark. Consequently, the greater the drawdown (DD), the more difficult it becomes to capture additional performance fee. In Derivatives terminology, the PM’s option becomes deeply “out-of-the-money” under substantial drawdowns. Like any other business, hedge funds must manage their cashflows properly. Excessive time spent below the high-watermark (TuW) means that the firm must subsist exclusively with inflows from the management fee, which typically decay as investors tend to redeem their shares if no new high-watermarks are achieved. This puts the firm at risk of not being able to meet financial obligations. Adequate stop-out procedures give investors the comfort that the firm is exerting proper due-diligence and complying with its fiduciary duty, by triggering stop-losses rather than allowing losses to compound.

A first goal of this paper is to provide an analytical framework to the stop-out problem, allowing for serial-conditionality of outcomes. We² show through Monte Carlo experiments that the closed-formula compact solution² presented here is accurate. In providing an analytical estimate of drawdown potential, even in the presence of serial-

¹ Our analysis can be applied to managers as well as strategies, and so we will refer to one or the other interchangeably.
² We use the term closed-formula solution in the sense of being exact and analytically derived. With the term compact, we mean that this solution does not involve mathematical operators such as discrete summation, and therefore it is amenable to Analysis.
conditionality, we open the possibility to integrate our results in optimization problems, rather than having to resort to computationally-expensive numerical methods. A second goal is to provide a theoretical justification as to why PMs with high Sharpe ratios may be given more permissive stop-out limits. A third goal is to formalize the relationship between the two main financial variables involved in a stop-out: DD and TuW. López de Prado and Peijan [2004] showed that serial correlation is the main feature responsible for extreme DD and TuW outcomes, even more so than Non-Normality. Non-Normality is a lesser concern because, as long as investment outcomes are independent and identically distributed (IID), the Central Limit Theorem ensures that the cumulative distribution of those investment outcomes converges to a Normal distribution as time passes. That paper allowed for serial-conditionality when modeling DD and TuW, however those values had to be computed through a Monte Carlo.

We begin our discussion with the standard mean-variance framework introduced by the seminal work of Markowitz [1952, 1956, 1959]. Under the assumption of IID Normal outcomes, we determine the maximum drawdown and the time under water associated with a particular confidence level. With these results as a backdrop, we generalize our framework to incorporate the possibility of first-order autoregressive (or AR(1)) investment outcomes. Allowing this type of serial conditionality offers a richer analysis than standard mean-variance approaches. Higher-order serial conditionality would lead to different numerical results, however not conceptually different conclusions than those derived from an AR(1) specification. This is because the key feature that leads to substantial DD and TuW is serial-dependence, which is already incorporated by an AR(1) process. To our knowledge, this is the first time that a compact expression is published which provides analytical estimates of drawdown potentials under first-order autoregressive cashflows.

This is not the first paper to discuss the impact that serial dependence has on performance metrics in general, or drawdown in particular. Lo [2002] derived the analytical solution to the asymptotic distribution of the Sharpe ratio under serial correlation, and found that the annual Sharpe ratio for a hedge fund can be overstated by as much as 65% because of the presence of serial correlation in monthly returns. In a very interesting study, Hayes [2006] fits a Markov chain (switching) model to hedge fund returns to estimate their drawdown potential under that type of serial dependence. Our work differs from Hayes’ in a number of ways. We use an autoregressive specification, which allows for Gaussian (continuous) shocks, while the Markov chain only considers a fixed step (up or down). Empirical evidence supports the autoregressive nature of hedge fund returns’ serial dependence. For example, Getmansky et al. [2004] study various sources for hedge fund returns’ serial correlation and conclude that exposure to illiquid investments is the most likely. It seems implausible that the returns of illiquid investments will follow the conditioned fixed step implied by a Markov chain. In his study, Hayes argues that a Markov chain asymptotically approximates to an AR(1) model, however he also acknowledges that the speed of convergence is slow, and both models will yield different answers. This is a problem, because we could not rely on a drawdown model which is only accurate after an extended time under water. The conclusion is that it would be
desirable to develop a drawdown model with serial correlation following an autoregressive specification. The present paper addresses this gap in the literature.

The rest of the study is organized as follows: Section 2 introduces the framework. Section 3 determines the maximum drawdown for a certain confidence level. Section 4 determines the maximum time under water for a given confidence level. Section 5 combines both concepts (maximum drawdown and maximum time under water) into the “triple penance rule.” Section 6 presents a numerical example. Section 7 explains why PMs with higher Sharpe ratios tend to receive less strict stop-out limits. Section 8 generalizes our framework to the case of first-order auto-correlated cashflows. Section 9 applies our framework to a long series of Hedge Fund Research (HFR) indices, and evaluates the impact that auto-correlation has on hedge funds’ downside potential and firing practices. Section 10 summarizes our conclusions. The mathematical appendices prove the propositions presented throughout the paper, and the Python code numerically validates the accuracy of our analytical results.

2.- THE FRAMEWORK
Suppose an investment strategy which yields a sequence of cash inflows $\Delta \pi_\tau$ as a result of a sequence of bets $\tau \in \{1, ..., \infty\}$, where

$$\Delta \pi_\tau = \mu + \sigma \varepsilon_\tau \tag{1}$$

such that the random shocks are IID distributed $\varepsilon_\tau \sim N(0,1)$. Because these random shocks $\varepsilon_\tau$ are independent and Normally distributed, so is the random variable $\Delta \pi_\tau$, with $\Delta \pi_\tau \sim N(\mu, \sigma^2)$. This provides for negative as well as positive outcomes, although in the context of stop-outs we are naturally interested in the former rather than the latter.

Drawdowns arise from the accumulation of cash inflows $\Delta \pi_\tau$ over $t$ sequential bets (or equivalently, a period of length $t$). If these bets are taken with a certain regular frequency, $t$ can also be interpreted in terms of time elapsed. For instance, twelve monthly bets would span a period of one calendar year. Let us define a function $\pi_t$ that accumulates the cashflows $\Delta \pi_\tau$ over $t$ bets.

$$\pi_t = \sum_{\tau=1}^{t} \Delta \pi_\tau \tag{2}$$

where $t \in \{0,1, ..., \infty\}$ and $\pi_0 = 0$. We set the cumulative performance to be zero at the origin, because we would like to evaluate the drawdown potential of $\pi_t$ at any arbitrary reset point $t=0$. In descriptive drawdown analysis, this reset point coincides with a high-watermark. Because $\pi_t$ is the aggregation of $t$ IID random variables $\Delta \pi_\tau \sim N(\mu, \sigma^2)$, we know that $\pi_t \sim N(\mu t, \sigma^2 t)$.

For a significance level $\alpha < \frac{1}{2}$, we define the quantile function for $\pi_t$:...
where \( Z_\alpha \) is the critical value of the Standard Normal distribution associated with a probability \( \alpha \) of performing worse than \( Q_{\alpha,t} \), i.e. \( \alpha = \text{Prob}[\pi_t \leq Q_{\alpha,t}] \). Then, drawdown is defined as:

\[
DD_{\alpha,t} = \max\{0, -Q_{\alpha,t}\}
\]

(4)

Note that \( DD_{\alpha,1} \) coincides with the standard Value-at-Risk (VaR) for that investment at a \( (1 - \alpha) \) confidence level (see Jorion [2006] for a discussion of VaR).

3. MAXIMUM DRAWDOWN
VaR is a risk metric limited to a particular horizon, typically one step ahead. We will move beyond VaR by determining the maximum probabilistic loss, or worst drawdown, regardless of the number of bets (or time horizon) involved. In words, we would like to answer the question: Up to how much could a particular strategy lose with a given confidence level? Proposition 1 computes analytically the maximum drawdown for a given significance level (proved in the Appendix).

**PROPOSITION 1**: Assuming IID cashflows \( \Delta \pi_t \sim N(\mu, \sigma^2) \), and \( \mu > 0 \), the maximum drawdown associated with a significance level \( \alpha < \frac{1}{2} \) is

\[
\text{MaxDD}_\alpha = \frac{(Z_\alpha \sigma)^2}{4\mu}
\]

(5)

which occurs at the time (or bet)

\[
t^*_\alpha = \left(\frac{Z_\alpha \sigma}{2\mu}\right)^2
\]

(6)

4. MAXIMUM TIME UNDER WATER
For a particular sequence \( \{\pi_t\} \), the time under water (\( TuW \)) is the minimum number of observations, \( t > 0 \), such that \( \pi_{t-1} < 0 \) and \( \pi_t \geq 0 \). We can determine its upper boundary for a significance level \( \alpha < \frac{1}{2} \) as the value \( t > 0 \) such that \( DD_{\alpha,t} = 0 \). Proposition 2 computes analytically the maximum time under water for a given significance level (proved in the Appendix).

**PROPOSITION 2**: Assuming IID cashflows \( \Delta \pi_t \sim N(\mu, \sigma^2) \), and \( \mu > 0 \), the maximum time under water associated with a significance level \( \alpha < \frac{1}{2} \) is
MaxTuW_\alpha = \left( \frac{Z_\alpha \sigma}{\mu} \right)^2 \quad (7)

Suppose that a PM experiences a performance \( \tilde{\pi}_t < 0 \) after \( t \) observations. For how long may we not be able to receive performance fee due to \( \tilde{\pi}_t \)? \( \tilde{\pi}_t \) is a realized performance, which is consistent with a quantile loss for some confidence level \( \tilde{\alpha}, -Q_{\tilde{\alpha},t} \). It would be useful to translate that loss \( \tilde{\pi}_t \) in terms of time under water, because that would allow us to express that monetary loss as a cost of opportunity (lost performance fee). Proposition 3 computes what is the maximum time under water implied by \( \tilde{\pi}_t \) (proved in the Appendix).

**PROPOSITION 3:** Given a realized performance \( \tilde{\pi}_t < 0 \) and assuming \( \mu > 0 \), the implied maximum time under water is

\[
MaxTuW_{\tilde{\pi}_t} = \frac{\tilde{\pi}_t^2}{\mu^2 t} - 2 \frac{\tilde{\pi}_t}{\mu} + t \quad (8)
\]

Proposition 3 is useful because, given that \( \tilde{\pi}_t \) has occurred, \( MaxTuW_{\tilde{\pi}_t} \) has become a realistic scenario of time under water. If, for example, \( \tilde{\pi}_t \) is so negative that \( MaxTuW_{\tilde{\pi}_t} > MaxTuW_\alpha \), the firm has a strong argument to stop-out the strategy, even if \( \tilde{\pi}_t > -MaxDD_\alpha \). Eq. (8) makes another key point: It not only matters how much money a PM has lost, but critically, for how long.

As we will see in Section 6, this implied maximum time under water is a better way of communicating stop-outs than giving a \( MaxDD_\alpha \) limit, because it allows us to enforce limits at all times, even before hitting the maximum admissible loss or exhausting the maximum time under water. Moreover, note that the calculation of \( MaxTuW_{\tilde{\pi}_t} \) only requires three input variables \( (\tilde{\pi}_t, t, \mu) \), where the first two are directly observable. Unlike in the case of \( MaxTuW_\alpha \), there is no need to input \( Z_\alpha \) or \( \sigma \).

5.- THE TRIPLE PENANCE RULE

The concepts of maximum drawdown \( (MaxDD_\alpha) \) and maximum time under water \( (MaxTuW_\alpha) \) are closely related. This is formalized in the following theorem (proved in the Appendix).

**THEOREM 1** (or “triple penance rule”): Under standard portfolio theory assumptions, a strategy’s maximum drawdown \( MaxDD_\alpha \) for a significance level \( \alpha \) occurs after \( t^*_\alpha \) observations. Then, the strategy is expected to remain under water for an additional \( 3t^*_\alpha \) after the maximum drawdown, with a confidence \( (1 - \alpha) \).

If we define \( Penance = \frac{MaxTuW_\alpha}{t^*_\alpha} - 1 \), then the “triple penance rule” tells us that, assuming independent \( \Delta \pi_t \) identically distributed as Normal (which is the standard portfolio theory assumption), \( Penance = 3 \), regardless of the Sharpe ratio of the
strategy. In other words, it takes three time longer to recover from the maximum drawdown than the time it took to produce it, for a given significance level $\alpha < \frac{1}{2}$. This rule has important practical implications with regards to how long it will take for a PM to recover from a fresh new bottom. Figure 1 provides a graphical representation.

[FIGURE 1 HERE]

Should $\Delta \pi_t$ exhibit positive serial correlation, $MaxDD_{\alpha}$, $t^*_\alpha$ and $MaxTuW_{\alpha}$ will tend to be substantially greater than in the case of $\Delta \pi_t$, IID Normal, however Penance will tend to be smaller than 3. We will discuss this case in Sections 8 and 9.

6.- NUMERICAL EXAMPLE
Consider two portfolio managers: PM1 and PM2. PM1 is expected to make US$10m over a year, with an annual standard deviation of also US$10m and a monthly trading frequency. For simplicity, we will assume a risk-free rate of zero.\(^3\) This implies an annualized Sharpe ratio of 1 (see Sharpe [1975, 1994] for a formal definition). On the other hand, PM2 will run the same risk budget as PM1, however he is expected to deliver a Sharpe ratio of 1.5. Table 1 summarizes this problem.

[TABLE 1 HERE]

For monthly bets, and a 95% confidence level, we would stop-out PM1 if he hits a cumulative loss of US$6,763,858.64 from his high-watermark, or stays under water for more than 2.706 years (about 33 months). Because PM2 is supposed to deliver a greater risk-adjusted performance (due to his Sharpe ratio of 1.5), he must trade under tighter constraints: We would stop-out PM2 if his losses exceed US$4,509,239.09, or if he remains under water longer than 1.2 years (about 15 months). Figure 2 plots the expected drawdown for a 95% confidence level, as a function of years passed since the last high-watermark, assuming monthly bets. In both cases, the time under water that follows the maximum drawdown is precisely 3 times the number of observations that occur up to the bottom performance. This is consistent with the “triple penance” rule.

[FIGURE 2 HERE]

Beyond stopping out in terms of maximum drawdown and time under water, this framework provides a basis for reassessing investments that perform worse than the quantile lines plotted in Figure 2 at any particular point in time. For example, suppose that PM1 has a cumulative loss of US$5,000,000 after being 2 years under water. Even though the loss is below the maximum drawdown, this scenario augurs a time under water of 3.125 years for the same confidence level implied by this observation (applying Proposition 3). That exceeds the pre-established limit of 2.706 years under water, and the firm may decide to stop-out PM1. Therefore, an effective way to communicate a

\(^3\) We do this to simplify calculations, without loss of generality. Alternatively, the reader could think of these performance numbers as net of the risk-free rate.
drawdown limit is to translate a realized loss in terms of the implied maximum time under water: Should \( MaxTuW_{\tilde{\alpha}} > MaxTuW_{\alpha} \), the strategy or PM will be stopped-out, regardless of the actual \( \tilde{\alpha} \), because the cost of opportunity (lost performance fee) is just too high for the firm.

7.- WHY DO BETTER MANAGERS GET LESS STRICT STOP-OUT LIMITS?

The previous numerical example gave a tighter stop-out limit to the PM with greater Sharpe ratio. And yet, the experienced reader is likely aware that hedge funds typically give greater stop-out limits to PMs with higher Sharpe ratios. To understand the reason for this apparent paradox, we need to incorporate into our model a bit of hedge fund business reality which is currently absent from our formal framework.

Hedge funds fund their operations through the collected management fee, and pay bonuses from the performance fee. Good PMs are likely to leave if they do not perceive a bonus within a certain timeframe. Hedge funds want to minimize the probability of defections among good PMs, who may abandon the firm leaving a loss behind. Thus, firms are willing to give more permissive stop-out limits to higher Sharpe ratio PMs.

Let us see how this argument fits in our framework. We can rewrite Eq. (6) as

\[
t^*_{\alpha} = T \left( \frac{Z_{\alpha}}{2SR} \right)^2
\]

where \( T \) is the total number of independent bets implemented in a year, and \( SR \) is the annualized Sharpe ratio. Combining Eq. (9) in Eq. (7), we obtain that

\[
\frac{MaxTuW_{\alpha}}{T} = \left( \frac{Z_{\alpha}}{SR} \right)^2
\]

\( \frac{MaxTuW_{\alpha}}{T} \) is the maximum time under water, for a confidence level \( 1 - \alpha \), expressed in years. What is new in this Section is that we are arguing that \( \frac{MaxTuW_{\alpha}}{T} \) is exogenously set, with some value \( \tilde{\alpha} \). For instance, \( \tilde{\alpha} = 1 \) is consistent with the hedge fund aiming at being in the position of paying a bonus after one year, so that the PM stays. Under these circumstances, Eq. (10) leads to

\[
\tilde{\alpha} = \left( \frac{Z_{\alpha}}{SR} \right)^2
\]

Eq. (11) evidences the existence of a trade-off between greater Sharpe ratio and greater tolerance to drawdowns. For double \( SR \), a double \( Z_{\alpha} \) is admissible, which allows for a substantially lower value of \( \alpha \) (recall that \( \alpha < \frac{1}{2} \), and thus \( Z_{\alpha} < 0 \)). More precisely, the significance level admissible, subject to an exogenously set target \( \tilde{\alpha} \), is
\[
\bar{\alpha} = Z \left[-SR\sqrt{K}\right] \tag{12}
\]

Once again, the negative sign appears due to the fact that \(\alpha < \frac{1}{2}\). Eq. (12) gives us a nice expression, which incorporates the business reality we discussed at the beginning of this Section. It explicitly tells us that a hedge fund is more permissive with PMs with high Sharpe ratios, despite the fact that we should expect those same PMs to perform better and hence operate with lower drawdowns.

Following our example from the previous Section, \(\bar{\alpha}\) is the answer to the question: *What significance level is consistent with being able to pay a bonus at the end of a period \(\bar{K}\)?* Following the example presented in Section 6, for an exogenously set \(\bar{K} = 1\) (one year), PM1 would be stopped-out at a significance level \(\bar{\alpha}_1 = 0.1587\), and PM2 would be stopped-out at a significance level \(\bar{\alpha}_2 = 0.0668\). As Figure 3 shows, now we are imposing stricter stop-out limits on PM1 than on PM2, the opposite of what we saw in Section 6. This is how in fact hedge funds typically operate, constrained as they are by the reality of having to protect themselves against defections.

[FIGURE 3 HERE]

Similarly, for an exogenously set \(\bar{K} = 2\) (two years), PM1 would be stopped-out at a significance level \(\bar{\alpha}_1 = 0.0787\), and PM2 would be stopped-out at a significance level \(\bar{\alpha}_2 = 0.0169\). Figure 4 shows that, as a result of the longer period that PMs are expected to tolerate without a bonus, the hedge fund sets even more permissive stop-out limits on PM2 relative to PM1 than we saw in Figure 3.

[FIGURE 4 HERE]

This greater permissiveness towards PMs with higher Sharpe ratios is contrary to what standard portfolio theory would have predicted, and yet our framework shows that hedge funds operating in this way act rationally, in an attempt to minimize the risk of defection among their most talented portfolio managers.

8. **STOP-OUT LIMITS UNDER FIRST-ORDER AUTO-CORRELATED CASHFLOWS**

Suppose an investment strategy which yields a sequence of cash inflows \(\Delta \pi_\tau\) as a result of a sequence of bets \(\tau \in \{1, ..., \infty\}\), where

\[
\Delta \pi_\tau = (1 - \varphi)\mu + \varphi\Delta \pi_{\tau-1} + \sigma \varepsilon_\tau \tag{13}
\]

such that the random shocks are IID distributed \(\varepsilon_\tau \sim N(0,1)\). Eq. (13) is initialized by a seed value \(\Delta \pi_0\), which is not necessarily null. These random shocks \(\varepsilon_\tau\) follow an independent and identically distributed Gaussian process, however \(\Delta \pi_\tau\) is neither an independent nor an identically distributed process. This is due to the parameter \(\varphi\), which incorporates a first-order serial-correlation effect of auto-regressive form. Appendix 5
shows that a necessary and sufficient condition for $\Delta \pi_t$ to be stationary is that $\varphi \in (-1,1)$. In that case, the above process has an asymptotic expectation $\lim_{T \to \infty} E_0[\Delta \pi_T] = \mu$, and an asymptotic variance $\lim_{T \to \infty} V_0[\Delta \pi_T] = \sigma^2 / (1 - \varphi^2)$, where the zero subscript at $E_0$ and $V_0$ denote expectations formed at the origin ($\tau = 0$). If these bets are taken with a certain regular frequency, $t$ can also be interpreted in terms of time elapsed. For instance, twelve monthly bets would span a period of one year.

Like in Section 2, let us define a function $\pi_t$ that accumulates the cash inflows $\Delta \pi_t$ over time.

$$\pi_t = \sum_{\tau=1}^{t} \Delta \pi_{\tau} \quad (14)$$

where $t \in \{0,1,...,\infty\}$ and $\pi_0 = 0$. We set the cumulative performance to be zero at the origin, because we would like to evaluate the drawdown potential of $\pi_t$ at any arbitrary reset point $t=0$. The following Propositions are proved in the Appendix.

**PROPOSITION 4**: Under the stationarity condition $\varphi \in (-1,1)$, the conditional distribution of a cumulative function $\pi_t$ of a first-order auto-correlated random variable $\Delta \pi_T$ follows a Normal distribution with parameters:

$$\pi_t \sim N \left( \frac{\varphi^{t+1} - \varphi}{\varphi - 1} (\Delta \pi_0 - \mu) + \mu t, \frac{\sigma^2}{(\varphi - 1)^2} \left( \frac{\varphi^{2(t+1)} - 1}{\varphi^2 - 1} - 2 \frac{\varphi^{t+1} - 1}{\varphi - 1} + t + 1 \right) \right) \quad (15)$$

**PROPOSITION 5**: The distribution of $\pi_t$ is non-stationary and unconditionally non-Normal.

For a significance level $\alpha < \frac{1}{2}$, we can estimate the drawdown function as the lower band for $\pi_t$ after $t$ bets:

$$Q_{\alpha,t} = \frac{\varphi^{t+1} - \varphi}{\varphi - 1} (\Delta \pi_0 - \mu) + \mu t + Z_{\alpha} \frac{\sigma}{|\varphi - 1|} \left( \frac{\varphi^{2(t+1)} - 1}{\varphi^2 - 1} - 2 \frac{\varphi^{t+1} - 1}{\varphi - 1} + t + 1 \right)^{1/2} \quad (16)$$

where $Z_{\alpha}$ is the critical value of the Standard Normal distribution associated with a probability $\alpha$ of performing worse than $Q_{\alpha,t}$. Like before, the drawdown function is finally obtained as $DD_{\alpha,t} = \max\{0, -Q_{\alpha,t}\}$. 
We are not aware of previously published analytical estimates of drawdown potentials under first-order auto-correlated cashflows. Proposition 4 is particularly useful in practice, because it gives us that closed-form solution presented in Eq. (16), and allows us to enunciate Proposition 6 (see the Appendix for a proof).

**PROPOSITION 6:** For \( \mu > 0 \), \( Q_{\alpha, t} \) is unimodal, a global minimum exists \( (\text{Min} Q_{\alpha}) \) and \( \text{Max} DD_{\alpha} = \max\{0,-\text{Min} Q_{\alpha}\} \) can be computed.

Appendices 9 and 10 present algorithms to determine the maximum drawdown and time under water in this more general framework. These procedures can be easily integrated in optimization problems, such as portfolio optimization subject to drawdowns or time under water constraints under serial conditionality. This is relevant, because many times researchers are compelled to adopt the ubiquitous IID assumption solely for computational reasons, contrary to empirical evidence that would have advised them to apply an expression like Eq. (16).

**9.- DOWNSIDE POTENTIAL IN THE HEDGE FUND INDUSTRY**

We are ready to put into practice the theory introduced in the earlier sections. We have downloaded from Bloomberg a long series of monthly Net Asset Values (NAVs) for Hedge Fund Research Indices (HFR), and selected those series that go from January 1st 1990 to January 1st 2013. This gives us 265 data points for each of the indices listed in Table 2.

[**TABLE 2 HERE**]

NAVs do not follow a stationary process (see Meucci [2005] for a comprehensive discussion of this subject). In order to apply our framework, we first need to perform a logarithmic transformation on the NAVs. The maximum likelihood estimator of \( \varphi \) is \( \hat{\varphi} = \text{Cov}_0 [\Delta \tau, \Delta \pi_{\tau-1}] (\text{Cov}_0 [\Delta \tau_{\tau-1}, \Delta \pi_{\tau-1}])^{-1} \), where \( \Delta \pi_{\tau} \) is the series of first order differences on the log-NAVs at observation \( \tau \), and \( \text{Cov}_0 \) is the covariance operator. The zero subscript denotes expectations formed at the origin (\( \tau = 0 \)). Following Appendix 5, we estimate \( \mu \) as \( \hat{\mu} = \mu_\infty \), and \( \sigma \) as \( \hat{\sigma} = \sqrt{\sigma_\infty (1 - \varphi^2)} \), where \( \mu_\infty = \lim_{\tau \to \infty} E_0 [\Delta \pi_\tau] = \mu \) is the asymptotic expected value and \( \sigma_\infty^2 = \lim_{\tau \to \infty} V_0 [\Delta \pi_\tau] = \frac{\sigma^2}{1 - \varphi^2} \) is the asymptotic variance. \( \mu_\infty \) and \( \sigma_\infty^2 \) can be approximated by the large-sample estimates of mean and variance of cashflows. Once we have estimated the triplet \( (\hat{\mu}, \hat{\sigma}, \hat{\varphi}) \), we can compute the \( \text{Max} DD_{\alpha} \) and \( \text{Max TuW}_{\alpha} \) using the code in Appendix 11. We are assuming \( \alpha = 0.05 \) and \( \Delta \pi_0 = 0 \), but different scenarios can be simulated by changing the appropriate parameters in the code. Table 3 reports the results.

[**TABLE 3 HERE**]

As discussed in Appendix 8, the convergence of the procedure requires \( \hat{\mu} > 0 \) and \( \hat{\varphi} \in [0,1) \), which is consistent with 24 of the 26 cases listed above. As we can deduce
from the t-Stat values for $\hat{\phi}$ reported in Table 3, $\hat{\phi}$ is statistically significantly in 21 out of 26 cases at a 95% confidence level. Despite of this overwhelming empirical evidence, let us suppose that $\hat{\phi} = 0$ in all of the above cases. This is of course a misleading assumption, however it is very common for practitioners and academics to assume that returns are independent to avoid dealing with serial correlation. Table 4 reports the corresponding $MaxDD_\alpha$ and $MaxTuW_\alpha$ under that scenario.

TABLE 4 HERE

Results in Table 4 can be computed in two different ways: i) Running the code in Appendix 11 for $\hat{\phi} = 0$ or ii) applying Eqs. (5) and (7). The conclusion is that unrealistically assuming $\hat{\phi} = 0$ leads to a gross underestimation of the downside potential of hedge fund investments. For example, in the case of “HFRI RV: Fixed Income-Convertible Arbitrage Index” (Code “HFRICAI Index”), the maximum drawdown under the assumption of independence is only 3.79%, while considering first-order auto-correlation would yield 11.60%. This means that wrongly assuming independence leads to a 67% underestimation compared to taking into account first-order auto-correlation. Maximum time under water if we assume independence is only 21.28 (monthly) observations, while considering first-order auto-correlation would yield 74.42. This means that wrongly assuming independence leads to a 71% underestimation compared to taking into account first-order auto-correlation. Penance measures how long it takes to recover from the maximum drawdown, as a multiple of the time it took to reach the bottom. More precisely, $Penance = \frac{MaxTuW_\alpha}{t^*_a} - 1$. Table 4 reports that Penance for hedge fund indices ranges between 1.6 and 3. Although positive serial correlation leads to greater drawdowns, longer periods to reach the bottom ($t^*_a$) and longer periods under water, the Penance may be substantially smaller. In particular, Penance is smaller the higher $\phi$ (Phi) and the higher the ratio $\frac{\mu}{\sigma}$ (Mean divided by Sigma). Figure 5 plots Penance for hedge fund indices with various $\hat{\phi}$.

FIGURE 5 HERE

These results introduce two interesting implications: First, hedge fund strategies are much riskier than what could be derived from performance metrics that rely on the ubiquitous IID assumption, such as Sharpe ratio, Sortino ratio, Treynor ratio, Information ratio, etc. (see Bailey and López de Prado [2012] for a discussion). This leads to an over-allocation of capital by Markowitz-style approaches to hedge fund strategies. Second, PMs and strategies evaluated by those IID-based metrics are being stopped-out much earlier than it would be appropriate. A good PM running a strategy that delivers auto-correlated cashflows may be unnecessarily stopped-out because the firm assumed IID cashflows. This is a particularly bad decision, because one positive aspect about strategies with auto-correlated cashflows is that their Penance is shorter than in the IID case.

We would like to understand whether hedge funds intending to accept a probability $\alpha_1$ of firing a truly skillful portfolio manager (a “false positive”) are effectively taking a different probability $\alpha_2$ as a result of assuming returns independence. Combining
Propositions 1 and 4 we can compute the $\alpha_2$ associated with $\pi_t = \text{MaxDD}_{\alpha_1} = \frac{(z_{\alpha_1}\sigma)^2}{4\mu}$ as

$$\alpha_2 = Z \left[ \frac{\frac{(z_{\alpha_1}\sigma)^2}{4\mu} - \frac{\varphi t_{\alpha_1}^*}{\varphi - 1} - \frac{\varphi (\Delta \pi_0 - \mu) + \mu t_{\alpha_1}^*}{(\varphi - 1)^2} \left( \frac{\varphi^2(t_{\alpha_1}^* + 1) - 1}{\varphi^2 - 1} - 2 \frac{\varphi t_{\alpha_1}^* + 1 - 1}{\varphi - 1} + t_{\alpha_1}^* + 1 \right)}{\sigma^2} \right]$$

(17)

where $Z$ is the cdf of the standard Normal distribution.\(^4\) Table 5 reports the effective proportion $\alpha_2$ of truly skillful portfolio managers fired by hedge funds, despite of aiming at a proportion $\alpha_1 = 0.05$. Again, this discrepancy arises because hedge funds assuming returns independence aim at a proportion $\alpha_1$, however they effectively get a proportion $\alpha_2$ because that assumption was false in most cases (see t-Stat values for $\varphi$ in Table 3).

[TABLE 5 HERE]

For all hedge fund styles, $\alpha_2 > \alpha_1$, which means that they are effectively firing a greater proportion of truly skillful portfolio managers than they originally intended. That proportion of over-firings is $\alpha_2 - \alpha_1$. Most hedge funds evaluate performance through traditional metrics, such as the Sharpe ratio, which assumes returns independence and would lead to the over-firing reported in Table 5. For example, hedge funds similar to those in the “HFRI RV: Fixed Income-Convertible Arbitrage Index” (code “HFRICAI Index”) may be firing 3.38 times (0.1688 vs. 0.05) the number of truly skillful portfolio managers, compared to the number they were willing to accept under the assumption of returns independence. Skillful managers that are fired by mistake need to be replaced in order to preserve performance, which increases personnel turnover. Our framework explains how the excessive turnover experienced by some hedge funds may be the result of unrealistically expecting their portfolio managers to deliver independent returns.

10.- CONCLUSIONS

Following standard portfolio theory assumptions, we have computed analytically the maximum drawdown and maximum time under water for a certain confidence level. We have shown how these concepts are intimately related through the “triple penance” rule. This rule states that, under standard portfolio theory assumptions, it takes three times longer to recover from the expected maximum drawdown than the time it takes to produce it, with the same confidence level. We have introduced a new drawdown concept called *Penance*, which measures how long it takes to recover from the maximum drawdown, as a multiple of the time it took to reach the bottom. We have also

\(^4\) Incidentally, Eq. (17) can be used to compute $\text{MaxTuW}_{\varphi}$ (Proposition 3) in the more general framework of first-order serial correlation. In order to do that, we simply have to input this $\varphi$ in the algorithm described in Appendix 10. For the reasons argued in Section 4, this would be a more effective way to communicate stop-out limits.
demonstrated an effective way to communicate drawdown limits, via an implied maximum time under water formulation.

According to this framework, for a certain confidence level, we should expect tighter stop-out limits imposed on portfolio managers with higher Sharpe ratios. That is rarely the case in practice. We have provided a theoretical justification to this observation, by recognizing that hedge funds must confront the risk of defection. They know that, should a good portfolio manager not receive a bonus within a certain period of time, he may try his luck at another firm and leave a loss behind. The consequence is that hedge funds assign greater confidence levels to portfolio managers with greater Sharpe ratios, and as a result they are quicker in stopping out portfolio managers with lower Sharpe ratios, just the opposite to what standard portfolio theory would have predicted.

We have complemented our study with a generalization of our framework to deal with the case of first-order auto-correlated cashflows. We derived a closed-formula compact expression which estimates the drawdown potential of a strategy without having to assume IID random shocks. An empirical study of hedge fund indices reveals that ignoring the effect of serial correlation leads to a gross underestimation of the downside potential of hedge fund strategies, by as much as 70%. Although positive auto-correlation leads to greater drawdowns, longer periods to reach the bottom and longer periods under water, the **Penance** may be substantially smaller. We find that some hedge funds may be firing more than three times the number of skillful portfolio managers, compared to the number that they were willing to accept, as a result of evaluating their performance through traditional metrics, such as the Sharpe ratio. The excessive turnover experienced by some hedge funds may be the result of unrealistically expecting their portfolio managers to deliver independent returns.

We are aware that researchers have been compelled to adopt the IID assumption in past, in disregard of contradicting empirical evidence, solely for computational reasons. We hope that the expression we have derived in this paper will allow them to take serial correlation into account in risk management, portfolio optimization and capital allocation applications. The Python code included in the Appendix numerically confirms the accuracy of our solution.
APPENDICES

A.1.- PROOF TO PROPOSITION 1

The drawdown function is defined as $DD_\alpha = \max\{0, -Q_\alpha\}$. We can compute its maximum as $\text{Max}DD_\alpha = \max\{0, -\text{Min}Q_\alpha\}$, where $\text{Min}Q_\alpha$ is the minimum value of the quantile function for a significance level $\alpha$. Next, we derive the expression for $\text{Min}Q_\alpha$.

We have seen that, in the case of independent and identically distributed Normal cashflows $\Delta \pi_t \sim N(\mu, \sigma^2)$, the quantile function is given by the expression:

$$Q_{\alpha,t} = \mu t + Z_\alpha \sigma \sqrt{t}$$  \hspace{1cm} (18)

The maximum drawdown is $\text{Max}DD_\alpha = \max\{0, -\text{Min}Q_\alpha\}$. Differentiating Eq. (18), the first order necessary condition that determines the globally and unconstrained minimum value for $Q_{\alpha,t}$ is given by:

$$\frac{\partial Q_{\alpha,t}}{\partial t} = \mu + \frac{1}{2\sqrt{t}} Z_\alpha \sigma = 0$$  \hspace{1cm} (19)

$$\frac{1}{2\sqrt{t}} Z_\alpha \sigma < 0 \text{ because } \alpha < \frac{1}{2} \iff Z_\alpha < 0,$$

thus this first order condition requires $\mu > 0$. Solving for $t$, we obtain the number of observations at which the lowest value of the function $Q_{\alpha,t}$ is realized,

$$t_\alpha^* = \left(\frac{Z_\alpha \sigma}{2\mu}\right)^2$$  \hspace{1cm} (20)

The second order sufficient condition is verified in $\frac{\partial^2 Q_{\alpha,t}}{\partial t^2} = -\frac{1}{4} Z_\alpha \sigma t^{-3/2} > 0$. This second derivative of the quantile function with respect to $t$ is strictly positive. This means that $Q_{\alpha,t}$ is convex with respect to $t$, which guarantees the existence of a global minimum.

Combining both conditions, we can then evaluate $Q_{\alpha,t}$ at that optimized value $t_\alpha^*$ to obtain its minimum value with a significance level $\alpha$:

$$\text{Min}Q_\alpha = Q_{\alpha,t^*} = \theta \left(\frac{Z_\alpha \sigma}{2\mu}\right)^2 + Z_\alpha \sigma \left|\frac{Z_\alpha \sigma}{2\mu}\right|^2 = -\frac{(Z_\alpha \sigma)^2}{4\mu}$$  \hspace{1cm} (21)

As expected, $\text{Min}Q_\alpha$ is not a function of $t$. Its negative value appears because, as we saw, $Z_\alpha < 0$ and $\mu > 0$ are sufficient conditions for the global maximum drawdown to exist.■

---

5 We will treat $t$ as a continuous variable in $\mathbb{R}^+$ for the purpose of differentiation.
A.2.- PROOF TO PROPOSITION 2
The time under water (TuW) is the number of observations (in terms of bets or time), \( t > 0 \), that elapse until we first observe that \( \pi_{t-1} < 0 \) and \( \pi_t \geq 0 \). We can determine its upper boundary for a significance level \( \alpha < \frac{1}{2} \) as the value \( t > 0 \) such that \( Q_{\alpha,t} = DD_{\alpha,t} = 0 \). This condition is satisfied at

\[
\mu t + Z_\alpha \sigma \sqrt{t} = t (\mu \sqrt{t} + Z_\alpha \sigma) = 0
\]

and because \( t > 0 \Rightarrow \mu \sqrt{t} + Z_\alpha \sigma = 0 \), we obtain

\[
MaxTuW_\alpha = \left( \frac{Z_\alpha \sigma}{\mu} \right)^2
\]

A.3.- PROOF TO PROPOSITION 3
Suppose that a portfolio manager experiences a cumulative performance at time \( t \) (since the last high-watermark) for an amount \( \bar{\pi}_t \), where \( \bar{\pi}_t < \pi_0 = 0 \). We can determine the implied significance level, \( \bar{\alpha} \), that is associated with such a performance after \( t \) independent observations. More precisely, we would like to compute the value of \( \bar{\alpha} \) that verifies

\[
\bar{\pi}_t = Q_{\bar{\alpha},t}
\]

Applying Eq. (18) on Eq. (24), we can solve for \( \bar{\alpha} \) as follows

\[
\bar{\alpha} = Z \left[ \frac{\bar{\pi}_t - \mu t}{\sigma \sqrt{t}} \right]
\]

where \( Z \) is the cumulative distribution function for the Standard Normal distribution. \( Z \) is a function and should not be confused with a critical value \( Z_\alpha \). For the sake of clarity, note that \( \alpha = Z[Z_\alpha], \forall \alpha \in (0,1) \).

Eq. (25) tells us that, for any observed performance, we can compute its implied statistical significance, which can be understood as \( \text{Prob}[\pi_t \leq \bar{\pi}_t] \). Hence, \( \bar{\alpha} \) can be interpreted as the ex-ante probability that this strategy would have performed below \( \bar{\pi}_t \) after \( t \) observations. Inserting Eq. (25) in Eq. (23), we obtain

\[
MaxTuW_{\bar{\alpha}} = \frac{\bar{\pi}_t^2}{\mu^2 t} - 2 \frac{\bar{\pi}_t}{\mu} + t
\]

Eq. (26) gives us the maximum time under water associated with the realized performance \( \bar{\pi}_t \), where \( \bar{\alpha} \) is implied by that same realized performance, \( \bar{\pi}_t \). This proposition allows us to communicate stop-out limits more effectively that a mere
MaxDD\_\alpha limit, because we do not need to wait until the maximum drawdown or time under water is reached. It suffices that MaxTuW_{\tilde{\alpha}} > MaxTuW_{\alpha} for any observed \tilde{\alpha}.\blacksquare

**A.4.- PROOF TO THEOREM 1 (“TRIPLE Penance Rule”)**

Comparing Eqs. (20) and (23), we derive the expression

\[ t^*_\alpha = \frac{1}{4} MaxTuW_{\alpha} \]  

(27)

Eq. (27) is true for any value \( \alpha \in \left( 0, \frac{1}{2} \right) \). These boundaries for \( \alpha \) first appeared in Eq. (19). We call it “triple penance rule” because it tells us that, in the case of independent and identically distributed normal cashflows \( \Delta \pi_t \), it takes three times longer to recover from the expected maximum drawdown than the time it took to produce it, for the same significance level \( \alpha < \frac{1}{2} \).\blacksquare

**A.5.- PROOF TO PROPOSITION 4**

This proposition generalizes the framework discussed in Section 2, by deriving the distribution of a cumulative function of a first-order auto-correlated random variable. Suppose an investment strategy which yields a sequence of cash inflows \( \Delta \pi_t \) as a result of a sequence of bets \( \tau \in \{1, \ldots, \infty\} \), where

\[ \Delta \pi_t = (1 - \varphi)\mu + \varphi \Delta \pi_{\tau-1} + \sigma \varepsilon_t \]  

(28)

such that the random shocks are IID distributed as \( \varepsilon_t \sim N(0,1) \). Recursively replacing the previous expression \( j \) times leads to

\[ \Delta \pi_t = (1 - \varphi)\mu \sum_{i=0}^{j-1} \varphi^i + \varphi^j \Delta \pi_{\tau-j} + \sigma \sum_{i=0}^{j-1} \varphi^i \varepsilon_{\tau-i} \]  

(29)

For a process initialized at \( \Delta \pi_0 \), we can extend the recursion back to the origin, in which case \( j = \tau \), and Eq. (29) becomes

\[ \Delta \pi_t = (1 - \varphi)\mu \sum_{i=0}^{\tau-1} \varphi^i + \varphi^\tau \Delta \pi_0 + \sigma \sum_{i=0}^{\tau-1} \varphi^i \varepsilon_{\tau-i} \]  

(30)

Eq. (30) tells us that \( \Delta \pi_t \) can also be interpreted as a moving average process of order \( \tau \). This expression evidences that \( \Delta \pi_t \) is a linear function of independent Gaussian random variables, hence \( \Delta \pi_t \) is also Gaussian (Grinstead and Snell [1997]). From Eq. (30), we can derive its mean and variance

\[ E_0[\Delta \pi_t] = (1 - \varphi)\mu \sum_{i=0}^{\tau-1} \varphi^i + \varphi^\tau \Delta \pi_0 \]  

(31)
\[ V_0[\Delta \pi_t] = \sigma^2 \sum_{l=0}^{T-1} \phi^{2l} \]

Eq. (31) shows that a necessary and sufficient condition for \( \Delta \pi_t \) to be stationary is that \( \phi \in (-1,1) \), in which case the above mean and variance asymptotically converge to 
\[ \lim_{T \to \infty} E_0[\Delta \pi_t] = \mu \quad \text{and} \quad \lim_{T \to \infty} V_0[\Delta \pi_t] = \frac{\sigma^2}{1-\phi^2}. \] From Eq. (31) we obtain that
\[
\Delta \pi_t \sim N \left( (1 - \phi) \mu \sum_{l=0}^{T-1} \phi^l, \sigma^2 \sum_{l=0}^{T-1} \phi^{2l} \right) \tag{32}
\]

Eq. (32) is a standard result of the time series analysis literature (see Hamilton [1990], for example). It is not immediately useful to us for two reasons. First, we are interested in the distribution of the cumulative process \( \pi_t = \sum_{\tau=1}^{t} \Delta \pi_\tau \), because drawdowns are not defined on \( \Delta \pi_t \). A VaR approach is typically interested in \( \Delta \pi_t \), but ours is a drawdown approach, which requires the modeling of the cumulative process. Second, Eq. (32) is not amenable to optimization because of its summation operators. Let us turn our attention to the cumulative process, \( \pi_t \). From Eq. (30):
\[
\pi_t = \sum_{\tau=1}^{t} \Delta \pi_\tau = (1 - \phi) \mu \sum_{\tau=1}^{t} \sum_{l=0}^{T-1} \phi^l + \Delta \pi_0 \sum_{\tau=1}^{t} \phi^\tau + \sigma \sum_{\tau=1}^{t} \sum_{l=0}^{T-1} \phi^l \varepsilon_{\tau-l} \tag{33}
\]

The last term can be conveniently operated as
\[
E_0 \left[ \sigma \sum_{\tau=1}^{t} \sum_{l=0}^{T-1} \phi^l \varepsilon_{\tau-l} \right] = 0 \tag{34}
\]
\[
V_0 \left[ \sigma \sum_{\tau=1}^{t} \sum_{l=0}^{T-1} \phi^l \varepsilon_{\tau-l} \right] = \sigma^2 V_0 \left[ \sum_{\tau=1}^{t} \left( \varepsilon_\tau \left( \sum_{l=0}^{T-1} \phi^l \right) \right) \right]
\]

We conclude that \( \pi_t \) is also a linear function of independent Gaussian random variables. This means that \( \pi_t \) is also Gaussian. We can compute the mean and variance of \( \pi_t \) as:
\[
E_0[\pi_t] = (1 - \phi) \mu \sum_{\tau=1}^{t} \sum_{l=0}^{T-1} \phi^l + \Delta \pi_0 \sum_{\tau=1}^{t} \phi^\tau \tag{35}
\]
\[
V_0[\pi_t] = \sigma^2 \sum_{\tau=1}^{t} \left( \sum_{l=0}^{T-1} \phi^l \right)^2
\]

Eq. (35) addresses the first limitation we discussed earlier (these are the moments on \( \pi_t \), not \( \Delta \pi_t \)), however it is still unsatisfactory with respect to the second feature: In order to
analyze the expressions in Eq. (35), we need to compute their compact form, i.e. excluding the summation operators that involve \( t \) and \( \tau \). Applying the Theorem of Geometric Series on \( E_0[\pi_t] \), we obtain:

\[
\sum_{\tau=1}^{t} \sum_{i=0}^{\tau-1} \phi^i = \sum_{\tau=1}^{t} \frac{\phi^\tau - 1}{\phi - 1} = \frac{1}{\phi - 1} \left(-t + \sum_{\tau=1}^{t} \phi^\tau\right)
\]

\[
= \frac{1}{\phi - 1} \left(-t + \frac{\phi^{t+1} - 1}{\phi - 1} - 1\right) = \frac{\phi^{t+1} - 1}{(\phi - 1)^2} - \frac{t + 1}{\phi - 1}
\]

Thus,

\[
E_0[\pi_t] = \frac{\phi^{t+1} - \phi}{\phi - 1} (\Delta \tau_0 - \mu) + \mu t
\]

This new expression for \( E_0[\pi_t] \) is easier to deal with from an Analysis perspective. Likewise, we can reduce \( V_0[\pi_t] \) by recognizing that

\[
\left(\sum_{i=0}^{t-\tau} \phi^i\right)^2 = \frac{\phi^{2(t-\tau+1)} - 2\phi^{t-\tau+1} + 1}{(\phi - 1)^2}
\]

\[
\sum_{\tau=1}^{t} \left(\sum_{i=0}^{t-\tau} \phi^i\right)^2 = \frac{1}{(\phi - 1)^2} \left(\sum_{\tau=1}^{t} \phi^{2(t-\tau+1)} - 2\sum_{\tau=1}^{t} \phi^{t-\tau+1} + t\right)
\]

\[
= \frac{1}{(\phi - 1)^2} \left(\phi^{2(t+1)} - 1 - 2\frac{\phi^{t+1} - 1}{\phi - 1} + t + 1\right)
\]

Thus,

\[
V_0[\pi_t] = \frac{\sigma^2}{(\phi - 1)^2} \left(\frac{\phi^{2(t+1)} - 1}{\phi^2 - 1} - 2\frac{\phi^{t+1} - 1}{\phi - 1} + t + 1\right)
\]

Finally, we conclude that
Eq. (40) has the two features we said we were looking for: First, it will allow us to compute a lower bound for the drawdown of a strategy with performance $\pi_t$. Second, we have been able to express this result in a compact form, amenable to Analysis.

A.6. - EXPERIMENTAL VERIFICATION OF PROPOSITION 4

The Python code contained in Snippet 1 provides numerical confirmation to the result derived analytically in Proposition 4. For example, one million simulations of a process with $\phi = \frac{1}{2}, \mu = 1, \sigma = 2, \Delta \pi_0 = 1, t = 25$ render an empirical distribution with a value of -6.7656 at the 5\textsuperscript{th}-percentile. The exact solution derived from Eq. (40) is -6.7816, implying a difference of only -0.016. Given any feasible combination of parameters, we can make the difference between the analytical and the Monte Carlo solutions as small as we wish, provided a sufficiently large number of runs.

```python
#!/usr/bin/env python
# On 20121226
# Monte Carlo to determine quantile
# by MLdP <lopezdeprado@lbl.gov>
from random import gauss
from numpy import zeros
from scipy.stats import scoreatpercentile,norm
#---------------------------------------------------------------
def main():
    #1) Parameters
    size=1e6 # size of the Monte Carlo experiment
    phi=.5 # AR(1) coefficient
    mu=1 # unconditional mean
    sigma=2 # Standard deviation of the random shock
    dPi0=1 # Bet at origin (initialization of AR(1))
bets=25 # Number of bets in the cumulative process
    confidence=.95 # Confidence level for quantile
    #2) Compute exact solution
    q=getQ(bets,phi,mu,sigma,dPi0,confidence)
    #3) Simulate Monte Carlo paths
    pnl=zeros(int(size))
    for i in range(int(size)): pnl[i]=getPath(phi,mu,sigma,dPi0,bets)
    #4) Compute empirical quantile, based on Monte Carlo paths
    q_mc=scoreatpercentile(pnl,(1-confidence)*100)
    #5) Report difference
print 'Quantile (Exact) = '+str(q)
```

Our code has been developed using the EPD 7.3 product (Enthought Python Distribution), which efficiently integrates all the necessary scientific libraries. For additional details, please visit https://www.enthought.com/products/epd.php.
print 'Quantile (Monte Carlo) = ' + str(q_mc)
print 'Difference = ' + str(q - q_mc)
return

def getQ(bets, phi, mu, sigma, dPi0, confidence):
    # Compute analytical solution to quantile
    # 1) Mean
    mean = (phi**((bets + 1) - phi)/(1 - phi) - mu) + mu * bets
    # 2) Variance
    var = sigma**2/(phi - 1)**2
    var *= (phi**((2*(bets + 1)) - 1)/(phi**2 - 1) - 2*(phi**((bets + 1)) - 1)/(phi - 1) + bets + 1)
    # 3) Quantile
    q = mean + norm.ppf(1 - confidence, 0, 1)*var**0.5
    return q

def getPath(phi, mu, sigma, dPi0, bets):
    # Run path for the Monte Carlo
    delta, pnl = dPi0, 0
    for i in range(int(bets)):
        delta = (1 - phi) * mu + gauss(0, 1) * sigma + delta * phi
        pnl += delta
    return pnl

if __name__ == '__main__': main()
where $Z_{\alpha}$ is the critical value of the Standard Normal distribution associated with a probability $\alpha$ of performing worse than $Q_{\alpha,t}$, i.e. $\alpha = \text{Prob}[\pi_t \leq Q_{\alpha,t}]$. Differentiating $Q_{\alpha,t}$ with respect to $t$:

$$\frac{\partial^i Q_{\alpha,t}}{\partial t^i} = \frac{\partial^i E_0[\pi_t]}{\partial t^i} + Z_{\alpha} \frac{\partial^i \sqrt{V_0[\pi_t]}}{\partial t^i} \quad (42)$$

where

$$\frac{\partial E_0[\pi_t]}{\partial t} = \frac{\ln[\phi] \theta^{t+1}}{\phi - 1} (\Delta \pi_0 - \mu) + \mu$$

$$\frac{\partial \sqrt{V_0[\pi_t]}}{\partial t} = \frac{\sigma}{2|\phi - 1|} \left( \frac{2 \ln[\phi] \phi^{2(t+1)}}{\phi^2 - 1} - \frac{\phi^{2(t+1)} - 1}{\phi^2 - 1} - 2 \frac{\phi^{t+1} - 1}{\phi - 1} + t + 1 \right)$$

$$- \frac{2 \ln[\phi] \phi^{t+1}}{\phi - 1} + 1 \right) \quad (43)$$

Note that these derivatives only exist for $\phi > 0$. That, combined with the stationarity condition, limits our range of study to $\phi \in (0,1)$. For that range of first-order autocorrelation, $\frac{2 \ln[\phi] \phi^{2(t+1)}}{\phi^2 - 1} - \frac{2 \ln[\phi] \phi^{t+1}}{\phi - 1} + 1 > 0$, thus $\frac{\partial \sqrt{V_0[\pi_t]}}{\partial t} > 0$. Risk is monotonically increasing over time, however the rate of increases converges to zero over time, as $\lim_{t \to \infty} \frac{\partial \sqrt{V_0[\pi_t]}}{\partial t} = 0^+$ (a convergence to zero from the right). $\frac{\partial E_0[\pi_t]}{\partial t}$ converges asymptotically to $\lim_{t \to \infty} \frac{\partial E_0[\pi_t]}{\partial t} = \mu$, starting below that level if $\Delta \pi_0 < \mu$, and starting above that level if $\Delta \pi_0 > \mu$. If $\mu$ is so large that at $t=1$ occurs that $\frac{\partial E_0[\pi_t]}{\partial t} > -Z_{\alpha} \frac{\partial \sqrt{V_0[\pi_t]}}{\partial t}$, $Q_{\alpha,t}$ will have its minimum at $Q_{\alpha,1}$ and $DD_\alpha = 0$. For a smaller but still positive $\mu$, $Q_{\alpha,t}$ will be initially a decreasing function of $t$, but it will eventually become an increasing function of $t$, once the effect of $\frac{\partial E_0[\pi_t]}{\partial t}$ overcomes the effect of $\frac{\partial \sqrt{V_0[\pi_t]}}{\partial t}$.

$$\alpha \in \left(0, \frac{1}{2}\right) \Rightarrow Z_{\alpha} < 0$$

so in order to guarantee a global minimum we would like to find that $\frac{\partial^2 E_0[\pi_t]}{\partial t^2} \geq 0$ and $\frac{\partial^2 \sqrt{V_0[\pi_t]}}{\partial t^2} < 0$. If we differentiate again the first expression in Eq. (43), we get:

\footnote{The case $\phi = 0$ would be easy to handle, by simplifying $\phi$ in $Q_{\alpha,t}$ before differentiating. However, we do not need to concern ourselves with an analysis of that case, since that was already addressed by Propositions 1 and 2.}
We can see that $\Delta \pi_0 < \mu \Rightarrow \frac{\partial^2 E_0[\pi_t]}{\partial t^2} > 0$, and $\Delta \pi_0 > \mu \Rightarrow \frac{\partial^2 E_0[\pi_t]}{\partial t^2} < 0$, but in any case this component wears out, since $\lim_{t \to \infty} \frac{\partial^2 E_0[\pi_t]}{\partial t^2} = 0$. If we differentiate again the second expression in Eq. (43), after some operations we get:

$$\frac{\partial^2 \sqrt{V_0[\pi_t]}}{\partial t^2} = \sigma (\ln[\varphi])^2 \left( \frac{2\varphi^{2(t+1)} - \varphi^{t+1}}{\varphi^2 - 1} \frac{\varphi^{t+1}}{\varphi - 1} \right)$$

$$\frac{|\varphi - 1| \sqrt{\frac{\varphi^{2(t+1)} - 1}{\varphi^2 - 1} - 2 \frac{\varphi^{t+1}}{\varphi - 1}} + t + 1}{4|\varphi - 1| \left( \frac{\varphi^{2(t+1)} - 1}{\varphi^2 - 1} - 2 \frac{\varphi^{t+1}}{\varphi - 1} + t + 1 \right)^{3/2}}$$

Because $\frac{2\varphi^{2(t+1)}}{\varphi^2 - 1} - \frac{\varphi^{t+1}}{\varphi - 1} > 0$ within the range $\varphi \in (0,1)$, $\frac{\partial^2 \sqrt{V_0[\pi_t]}}{\partial t^2}$ can be positive for a small $t$. So $Q_{\alpha,t}$ could be initially concave, either because of $\frac{\partial^2 E_0[\pi_t]}{\partial t^2} < 0$, or both. As $t$ increases, $\varphi^t \to 0$, and $\frac{\partial^2 \sqrt{V_0[\pi_t]}}{\partial t^2} \approx -\frac{\sigma}{4|\varphi - 1| \left( \frac{1}{\varphi^2 - 1} + 2 \frac{1}{\varphi - 1} + t + 1 \right)^{3/4}} < 0$ while $\frac{\partial^2 E_0[\pi_t]}{\partial t^2} \approx 0$, so in any case $Q_{\alpha,t}$ is convex after a sufficient number of bets. $Q_{\alpha,t}$ remains convex but increasingly linear, with $\lim_{t \to \infty} \frac{\partial^2 \sqrt{V_0[\pi_t]}}{\partial t^2} = 0^-$ (a convergence to zero from the left).

Putting all the pieces together, we know that $Q_{\alpha,t}$ could be initially a concave function of $t$, but eventually it becomes concave. It is guaranteed that $Q_{\alpha,t}$ has either zero or one inflexion point. As long as $\mu > 0$, $Q_{\alpha,t}$ is unimodal and a global minimum exists. The following section shows how to compute $\text{Max} DD_\alpha = \max \{0, -\text{Min} Q_{\alpha}\}$.

A.9.- ALGORITHM FOR FINDING THE MAXIMUM DRAWDOWN
Proposition 6 has proved that $Q_{\alpha,t}$ is a unimodal function of $t$. Using this property, Snippet 2 applies the Golden Section algorithm to find the maximum drawdown of the process described in Proposition 4.

```python
#!/usr/bin/env python
# On 20121230
# Get maximum drawdown
# by MLdP <lopezdeprado@lbl.gov>
from scipy.stats import norm
```
def main():
    #1) Parameters
    phi=.5  # AR(1) coefficient
    mu=1   # unconditional mean
    sigma=2  # Standard deviation of the random shock
    dPi0=1  # Bet at origin (initialization of AR(1))
    confidence=.95  # Confidence level for quantile
    #2) Compute MinQ
    t,minQ=getMinQ(phi,mu,sigma,dPi0,confidence)
    print 'MinQ = '+str(minQ)
    print 'Time at MinQ = '+str(t)
    print 'MaxDD = '+str(max(0,-minQ))
    return

def getMinQ(phi,mu,sigma,dPi0,confidence):
    # Compute MinQ
    q,bets=0,0
    #1) Determine extremes of search
    while not q>0:
        bets+=1
        q=getQ(bets,phi,mu,sigma,dPi0,confidence)
    #2) Compute min of q
    kargs={'args':(phi,mu,sigma,dPi0,confidence)}
    t,minQ=goldenSection(getQ,0,bets,**kargs)
    return t,minQ

def getQ(bets,phi,mu,sigma,dPi0,confidence):
    # Compute analytical solution to quantile
    #1) Mean
    mean=(phi**(bets+1)-phi)/(1-phi)*(dPi0-mu)+mu*bets
    #2) Variance
    var=sigma**2/(phi-1)**2
    var*=(phi***(2*(bets+1))-1)/(phi-1)**2-2*(phi**(bets+1)-1)/(phi-1)+bets+1
    #3) Quantile
    q=mean+norm.ppf(1-confidence,0,1)*var**.5
    return q

def goldenSection(obj,a,b,**kargs):
    # Golden section method. Maximum if kargs['minimum']==False is passed
    from math import log,ceil
tol,sign,args=1.0e-9,1,None
    if 'minimum' in kargs and kargs['minimum']==False:sign=-1
    if 'args' in kargs:args=kargs['args']
    numIter=int(ceil(-2.078087*log(tol/abs(b-a))))
    r=0.618033989
c=1.0-r
    # Initialize
    x1=r*a+c*b;x2=c*a+r*b
    f1=sign*obj(x1,*args);f2=sign*obj(x2,*args)
    # Loop
    for i in range(numIter):
        if f1>f2:
            a=x1
            x1=x2;f1=f2
            x2=c*a+r*b;f2=sign*obj(x2,*args)
A.10.- ALGORITHM FOR COMPUTING THE TIME UNDER WATER

Likewise, we can use the result obtained in Proposition 4 to find the time under water in the more general case of first-order serially correlated cashflows.

```python
# Get maximum time under water
# by MLdP <lopezdeprado@lbl.gov>
from scipy.stats import norm

# Boilerplate
if __name__ == '__main__':
    main()

def main():
    # 1) Parameters
    phi = .5  # AR(1) coefficient
    mu = 1   # unconditional mean
    sigma = 2  # Standard deviation of the random shock
    dPi0 = 1  # Bet at origin (initialization of AR(1))
    confidence = .95  # Confidence level for quantile

    # 2) Compute TuW
    tuw = getTuW(phi, mu, sigma, dPi0, confidence)
    print 'MaxTuW = ' + str(tuw)
    return

def getTuW(phi, mu, sigma, dPi0, confidence):
    # Compute TuW
    q, bets = 0, 0
    # 1) Determine extremes of search
    while not q > 0:
        bets += 1
        q = getQ(bets, phi, mu, sigma, dPi0, confidence)
    # 2) Compute root of q polynomial
    kargs = {'args': (phi, mu, sigma, dPi0, confidence)}
    tuw, q = goldenSection(diff, bets - 1, bets, **kargs)
    return tuw

def getQ(bets, phi, mu, sigma, dPi0, confidence):
    # Compute analytical solution to quantile
    # 1) Mean
    mean = (phi ** (bets + 1) - phi) / (1 - phi) * (dPi0 - mu) + mu * bets
    # 2) Variance
    var = sigma ** 2 / (phi - 1) ** 2
    var *= (phi / (2 * (bets + 1)) - 2 * (phi * (bets + 1) - 1) / (phi ** 2 - 1) - 2 * (phi ** 2 - 1) / (phi ** 2 - 1) - 2 * (phi ** 2 - 1) / (phi ** 2 - 1)) + bets + 1
    # 3) Quantile
```

Snippet 2 – Python code for finding the maximum drawdown associated with Proposition 4
q = mean + norm.ppf(1 - confidence, 0, 1) * var ** 0.5
return q
#---------------------------------------------------------------
def diff(bets, phi, mu, sigma, dPi0, confidence):
    return abs(getQ(bets, phi, mu, sigma, dPi0, confidence))
#---------------------------------------------------------------
def goldenSection(obj, a, b, **kargs):
    # Golden section method. Maximum if kargs['minimum'] == False is passed
    tol, sign, arg = 1.0e-9, 1, None
    if 'minimum' in kargs and kargs['minimum'] == False: sign = -1
    if 'args' in kargs: arg = kargs['args']

    numIter = int(ceil(-2.078087 * log(tol / abs(b - a))))
    r = 0.618033989
    c = 1.0 - r

    # Initialize
    x1 = r * a + c * b; x2 = c * a + r * b
    f1 = sign * obj(x1, *arg); f2 = sign * obj(x2, *arg)

    # Loop
    for i in range(numIter):
        if f1 > f2:
            a = x1
            x1 = x2; f1 = f2
            x2 = c * a + r * b; f2 = sign * obj(x2, *arg)
        else:
            b = x2
            x2 = x1; f2 = f1
            x1 = r * a + c * b; f1 = sign * obj(x1, *arg)

    if f1 < f2: return x1, sign * f1
    else: return x2, sign * f2
#---------------------------------------------------------------
# Boilerplate
if __name__ == '__main__': main()

Snippet 3 – Python code for finding the time under water associated with Proposition 4

A.11.- REPRODUCING OUR RESULTS
The empirical study in Section 9 can be reproduced running the code in Snippet 4. It requires the module in Appendix 9 to be stored as DD2.py and the module in Appendix 10 as DD3.py. Tables 3 and 4 can be used as input csv files by assigning their name to the variable inFileName. Their location is declared in path. The relevant columns are declared in fields, and the confidence level in confidence. By default, the program assumes that $\Delta \pi_0 = 0$, but that can be changed by assigning any other preferred value to the parameter dPi0. The output will be reported in a csv file named outFileName.
```python
try:
    float(input)
    return True
except:
    return False
#---------------------------------------------------------------
def main():
#1) Parameters
path='D:\Stop out/
inFileName='Data1.csv'
outFileName='Results1.csv'
fields=['Code','Mean','Phi','Sigma']
confidence=.95
dPi0=0
#2) Read file
inFile=open(path+inFileName,'r')
outFile=open(path+outFileName,'w')
headers=inFile.readline().split(',')
indices=[headers.index(i) for i in fields]
for line in inFile:
#3) Get Input
    params={}
    line=line[:-1].split(',
    for i in indices:
        if isNumber(line[i])==True:
            params[headers[i]]=float(line[i])
        else:
            params[headers[i]]=line[i]
#4) Compute MaxDD,MaxTuW
    if params['Mean']>=0 and params['Phi']>=0:
        t,minQ=DD2.getMinQ(params['Phi'],params['Mean'],
                          params['Sigma'],dPi0,confidence)
        maxDD=max(0,-minQ)
        maxTuW=DD3.getTuW(params['Phi'],params['Mean'],
                          params['Sigma'],dPi0,confidence)
    else:
        maxDD,t,maxTuW='--', '--', '--'
#5) Store result
    msg=params['Code']+',','+str(maxDD)+',','+str(t)+',','+str(maxTuW)
    outFile.writelines(msg+''
    print msg
return
#---------------------------------------------------------------
# Boilerplate
if __name__=='__main__':main()
```

Snippet 4 – Python code for reproducing our results in Tables 3 and 4
Table 1 – Trading parameters for evaluating the stop-out

<table>
<thead>
<tr>
<th>Parameters</th>
<th>PM1</th>
<th>PM2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected annual PnL</td>
<td>10,000,000</td>
<td>15,000,000</td>
</tr>
<tr>
<td>Expected Std annual PnL</td>
<td>10,000,000</td>
<td>10,000,000</td>
</tr>
<tr>
<td># Independent trades per year</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>Confidence</td>
<td>95%</td>
<td>95%</td>
</tr>
</tbody>
</table>

Table 2 – Selected Hedge Fund Research Indices

Table 2 lists the hedge fund indices in the HFR database with a history from 01/01/1990 to 01/01/2013. These are the indices that we have used in the empirical study presented in Section 9.
Table 3 reports the descriptive statistics computed on the hedge fund indices listed in Table 2, when we take into account first-order auto-correlation. \textit{MaxDD} is the maximum drawdown at a $\alpha = 0.05$ significance level, which occurs after $t^*$ observations. At that same significance level, these hedge fund indices remain under water for the period reported in column \textit{MaxTuW}. \textit{Penance} measures how long it takes to recover from the maximum drawdown as a multiple of the time it took to reach the bottom. As we can appreciate, \textit{Penance} ranges between 1.6 and 3, and it is smaller the higher $\phi$ (\textit{Phi}) and the higher the ratio $\mu/\sigma$ (\textit{Mean} divided by \textit{Sigma}). Although positive serial correlation leads to greater drawdowns, longer $t^*$ and longer periods under water, the \textit{Penance} is smaller.

<table>
<thead>
<tr>
<th>Code</th>
<th>Mean</th>
<th>StDev</th>
<th>Phi</th>
<th>Sigma</th>
<th>t-Stat(Phi)</th>
<th>MaxDD</th>
<th>t*</th>
<th>MaxTuW</th>
<th>Penance</th>
</tr>
</thead>
<tbody>
<tr>
<td>HFRIFOF Index</td>
<td>0.0055</td>
<td>0.0170</td>
<td>0.3594</td>
<td>0.0158</td>
<td>6.2461</td>
<td>6.65%</td>
<td>14.5551</td>
<td>52.1831</td>
<td>2.5852</td>
</tr>
<tr>
<td>HFRIFWI Index</td>
<td>0.0089</td>
<td>0.0202</td>
<td>0.3048</td>
<td>0.0192</td>
<td>5.1907</td>
<td>4.74%</td>
<td>7.3222</td>
<td>24.4918</td>
<td>2.3449</td>
</tr>
<tr>
<td>HFRIFHI Index</td>
<td>0.0089</td>
<td>0.0264</td>
<td>0.2651</td>
<td>0.0255</td>
<td>4.4601</td>
<td>7.27%</td>
<td>9.0236</td>
<td>32.1120</td>
<td>2.5587</td>
</tr>
<tr>
<td>HFRIMO Index</td>
<td>0.0095</td>
<td>0.0215</td>
<td>0.1844</td>
<td>0.0211</td>
<td>3.0419</td>
<td>4.15%</td>
<td>5.4157</td>
<td>19.1093</td>
<td>2.5285</td>
</tr>
<tr>
<td>HFRIFOFD Index</td>
<td>0.0052</td>
<td>0.0174</td>
<td>0.3535</td>
<td>0.0163</td>
<td>6.1295</td>
<td>7.52%</td>
<td>16.9638</td>
<td>61.9703</td>
<td>2.6531</td>
</tr>
<tr>
<td>HFRIDSI Index</td>
<td>0.0096</td>
<td>0.0188</td>
<td>0.5458</td>
<td>0.0158</td>
<td>10.5612</td>
<td>5.40%</td>
<td>10.7065</td>
<td>30.4208</td>
<td>1.8413</td>
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<tr>
<td>HFRICMNI Index</td>
<td>0.0052</td>
<td>0.0094</td>
<td>0.1644</td>
<td>0.0093</td>
<td>2.7035</td>
<td>1.33%</td>
<td>3.4722</td>
<td>11.6921</td>
<td>2.3674</td>
</tr>
<tr>
<td>HFRIFOC Index</td>
<td>0.0048</td>
<td>0.0116</td>
<td>0.4557</td>
<td>0.0103</td>
<td>8.3023</td>
<td>4.00%</td>
<td>11.9696</td>
<td>39.0229</td>
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<td>HFRIMDI Index</td>
<td>0.0095</td>
<td>0.0192</td>
<td>0.3916</td>
<td>0.0177</td>
<td>6.9021</td>
<td>4.34%</td>
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<td>-0.0188</td>
<td>0.0216</td>
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<td>--</td>
<td>--</td>
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<td>HFRIFIHY Index</td>
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<td>0.0177</td>
<td>0.4838</td>
<td>0.0155</td>
<td>9.8720</td>
<td>6.69%</td>
<td>13.3986</td>
<td>43.7383</td>
<td>2.2644</td>
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<td>0.0129</td>
<td>0.5059</td>
<td>0.0111</td>
<td>9.5874</td>
<td>3.12%</td>
<td>8.9080</td>
<td>25.0456</td>
<td>1.8116</td>
</tr>
<tr>
<td>HFRIRVA Index</td>
<td>0.0080</td>
<td>0.0130</td>
<td>0.4528</td>
<td>0.0116</td>
<td>8.2430</td>
<td>2.00%</td>
<td>5.9134</td>
<td>15.3920</td>
<td>1.6029</td>
</tr>
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<td>HFRIMAI Index</td>
<td>0.0071</td>
<td>0.0104</td>
<td>0.2982</td>
<td>0.0100</td>
<td>5.0670</td>
<td>1.08%</td>
<td>3.2508</td>
<td>8.9163</td>
<td>1.7423</td>
</tr>
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<td>HFRICAL Index</td>
<td>0.0071</td>
<td>0.0200</td>
<td>0.5780</td>
<td>0.0163</td>
<td>11.4685</td>
<td>11.60%</td>
<td>22.1308</td>
<td>74.4170</td>
<td>2.3626</td>
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<td>HFRIBAI Index</td>
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<td>0.0410</td>
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<td>21.71%</td>
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<td>5.3109</td>
<td>22.57%</td>
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<td>0.0907</td>
<td>0.0533</td>
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<td>18.2415</td>
<td>67.7961</td>
<td>2.7166</td>
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<td>HFRINHAI Index</td>
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<td>0.0367</td>
<td>0.2011</td>
<td>0.0359</td>
<td>3.3299</td>
<td>12.84%</td>
<td>13.8963</td>
<td>52.7651</td>
<td>2.7971</td>
</tr>
<tr>
<td>HFRIFWIG Index</td>
<td>0.0094</td>
<td>0.0360</td>
<td>0.2314</td>
<td>0.0350</td>
<td>3.8573</td>
<td>14.15%</td>
<td>16.4723</td>
<td>62.5481</td>
<td>2.7972</td>
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<td>HFRIFOFM Index</td>
<td>0.0056</td>
<td>0.0159</td>
<td>0.0422</td>
<td>0.0159</td>
<td>0.6842</td>
<td>3.25%</td>
<td>6.0074</td>
<td>23.5097</td>
<td>2.9135</td>
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<td>HFRIFWIC Index</td>
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<td>0.0390</td>
<td>0.0505</td>
<td>0.0390</td>
<td>0.8200</td>
<td>12.59%</td>
<td>14.3295</td>
<td>56.6921</td>
<td>2.9563</td>
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<td>0.0363</td>
<td>0.0954</td>
<td>0.0361</td>
<td>1.5542</td>
<td>12.55%</td>
<td>15.4084</td>
<td>60.4123</td>
<td>2.9207</td>
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<td>0.1608</td>
<td>0.0458</td>
<td>2.6428</td>
<td>17.61%</td>
<td>16.8089</td>
<td>65.0637</td>
<td>2.8708</td>
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</table>

Table 3 – Maximum Drawdown and Time under Water considering first-order serial correlation
It is common to assume returns independence (thus disregarding evidence of serial correlation) to simplify calculations. Table 4 reports the maximum drawdown ($MaxDD$), the observation at which the maximum drawdown occurs ($t^*$), and the maximum time under water ($MaxTuW$) for a significance level of $\alpha = 0.05$. Wrongly assuming $\varphi = 0$ (see column $Phi$) may lead to a gross underestimation of the downside potential, in some cases by as much as 70%. As predicted by the “triple penance rule”, the number of observations it takes to resurface after the maximum drawdown is exactly 3 times the number of observations it took to reach that maximum drawdown, with the same confidence level.

<table>
<thead>
<tr>
<th>Code</th>
<th>Mean</th>
<th>Phi</th>
<th>Sigma</th>
<th>$MaxDD$</th>
<th>$t^*$</th>
<th>$MaxTuW$</th>
<th>Penance</th>
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<tbody>
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<td>0.0055</td>
<td>0.0000</td>
<td>0.0170</td>
<td>3.53%</td>
<td>6.3996</td>
<td>25.5985</td>
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<td>HFRIFWI Index</td>
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<td>0.0000</td>
<td>0.0202</td>
<td>3.10%</td>
<td>3.4905</td>
<td>13.9621</td>
<td>3.0000</td>
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<tr>
<td>HFRIEHI Index</td>
<td>0.0099</td>
<td>0.0000</td>
<td>0.0264</td>
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<tr>
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<td>0.0215</td>
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<td>HFRIFOFD Index</td>
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<td>0.0174</td>
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<td>0.0000</td>
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<td>3.9492</td>
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<td>0.0192</td>
<td>2.63%</td>
<td>2.7554</td>
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<tr>
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<td>17.2870</td>
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<td>0.0177</td>
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<td>0.0000</td>
<td>0.0130</td>
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<td>0.0000</td>
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<td>8.9357</td>
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<td>0.0360</td>
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Table 4 – Maximum Drawdown and Time under Water ignoring serial correlation
Table 5 – Intended vs. actual probability of false positives

<table>
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<th>Code</th>
<th>MaxDD</th>
<th>t*</th>
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Table 5 reports the effective probability of false positives when portfolio managers or strategies are stopped-out based on drawdown limits that ignore first-order autocorrelation. For all hedge fund styles, the actual probability of false positives is considerably greater than the one intended. Because many firms evaluate their managers’ performance assuming independent returns (e.g., Sharpe ratio), they are improperly stopping them out. In some cases, they may be firing more than three times the number of skillful portfolio managers, compared to the number they were willing to accept under the (wrong) assumption of returns independence.
Figure 1 provides a graphical representation of the Triple Penance rule. It takes three time longer to recover from the maximum drawdown \((\text{MaxTuW}_\alpha)\) than the time it took to produce it \((t^*_\alpha)\), for a given significance level \(\alpha < \frac{1}{2}\), regardless of the PM’s Sharpe ratio.
Figure 2 plots the drawdown function $Q_{\alpha,t}$ as time passes, for $\alpha = 0.05$, where PM1 has an annualized Sharpe ratio of 1 (annual mean and standard deviation of US$10m), and PM2 has an annualized Sharpe ratio of 1.5 (annual mean of US$15m, and annual standard deviation of US$10m). For that 95% confidence level, PM1 reaches a maximum drawdown at US$6,763,858.64 after 0.676 years, and remains up to 2.706 years under water, whereas PM2 reaches a maximum drawdown at US$4,509,239.09 after 0.3 years, and remains 1.202 years under water. These results are consistent with the “triple penance” rule.
Figure 3 plots the drawdown function $Q_{\alpha,t}$ as time passes, where $\bar{\alpha}$ has been computed individually to meet the goal of being under water a maximum of 1 year. As a result, whereas before we had a tighter stop-out for the portfolio manager with higher Sharpe ratio, now the stricter stop-out is imposed on the portfolio manager with lower Sharpe ratio. This is consistent with the business reality that a hedge fund faces the risk of seeing good portfolio managers defecting if a performance bonus cannot be paid within a certain period.
Figure 4 – Drawdown and Time under water for PM1 and PM2, with confidence levels that aim at a maximum of 2 years under water

Figure 4 plots the drawdown function $Q_{\alpha,t}$ as time passes, where $\tilde{\alpha}$ has been computed to meet the goal of being under water a maximum of 2 years. As a result of the longer maximum period (2 years instead of 1), we are even more permissive of in setting stop-out levels for PM2 than we were in Figure 2.
Figure 5 plots Penance for hedge fund indices with various $\phi$. Although positive serial correlation leads to greater drawdowns, longer $t_{\alpha}$ and longer periods under water, the Penance may be substantially smaller. In particular, Penance is smaller the higher $\phi$ (Phi) and the higher the ratio $\frac{\mu}{\sigma}$ (Mean divided by Sigma).
REFERENCES


